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Discrete-time analogues of some nonlinear oscillators in the inverse-square potential

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Abstract. We propose a family of symplectic maps providing finite-difference approximations for the Wojciechowski and Rosochatius systems. The Lax pair representations are found and complete integrability is proved.

1. Introduction

Discrete-time analogues of completely integrable systems in classical mechanical have in the last few years attracted growing attention (see, e.g., [1] and references therein and below). They turn out to possess mathematical structures which are by no means simpler than the corresponding structures of their continuous-time counterparts. On the contrary, these discrete structures are often richer and more general. The number of examples of integrable multi-dimensional maps remains relatively small, however, and general procedures allowing one to unify all the known examples and to find new ones in a systematic way are still lacking.

Several families of the systems

$$x_k(t+h) + x_k(t-h) = f_k(x(t)) \qquad x = (x_1, \ldots, x_N)^{\mathrm{T}} \in \mathbb{R}^N \tag{1}$$

defining integrable multi-dimensional symplectic maps have been found: in particular, discrete-time analogues of certain one-dimensional oscillators [2], of the Neumann system describing the motion of a particle on a sphere in a harmonic potential [3], and of generalized Toda lattices [4], of the Garnier system and its generalizations related to Hermitian symmetric spaces [5].

In this paper we propose a kind of generalization of (1). Fix N real numbers m_k^2, $1 \leqslant k \leqslant N$ (so that m_k are either real or pure imaginary), and consider the system

$$q_k(t+h)\sqrt{1 - \frac{m_k^2}{q_k^2(t+h)q_k^2(t)}} + q_k(t-h)\sqrt{1 - \frac{m_k^2}{q_k^2(t)q_k^2(t-h)}} = f_k(q(t)) \qquad 1 \leqslant k \leqslant N \tag{2}$$

(we will also consider a more general case, when the right-hand side reads $f_k(q(t), q(t-h))$). In (2), as in (1), the functions $q(t), t \in h\mathbb{Z}$, are not supposed to be defined for all $t \in \mathbb{R}$. Unlike (1), it is natural to assume that all the q_k's are positive, so that

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$q = (q_1, \ldots, q_N)^{\mathrm{T}} \in \mathbb{R}_+^N$. One would like, as usual, to set $r(t) = q(t-h)$, in order for (2) to define a map $(q(t), r(t)) \mapsto (q(t+h), r(t+h))$ of $\mathbb{R}_+^N \times \mathbb{R}_+^N$ onto itself. Some remarks are in order, however. First, in the case of $m_k^2 > 0$ the domain of this map is not the whole $\mathbb{R}_+^N \times \mathbb{R}_+^N$, but its subset defined by $q_k^2 r_k^2 \geqslant m_k^2$. Second, this map is well defined if both the positive and negative branches of the square root are allowed. In fact 2^N different maps are encoded in (2): if one fixes the branches for

$$\sqrt{1 - \frac{m_k^2}{q_k^2(t) q_k^2(t-h)}} \qquad 1 \leqslant k \leqslant N$$

then the branches for

$$\sqrt{1 - \frac{m_k^2}{q_k^2(t+h) q_k^2(t)}} \qquad 1 \leqslant k \leqslant N$$

are uniquely defined by (2). So it might be more convenient to re-write equation (2) as

$$\varepsilon_k(t+h) q_k(t+h) \sqrt{1 - \frac{m_k^2}{q_k^2(t+h) q_k^2(t)}} + \varepsilon_k(t) q_k(t-h) \sqrt{1 - \frac{m_k^2}{q_k^2(t) q_k^2(t-h)}}$$
$$= f_k(q(t))$$

where $\varepsilon_k \in \{\pm 1\}$, $1 \leqslant k \leqslant N$, and all the square roots are now assumed to be positive. In this form the equation unambigously defines a map

$$(q(t), r(t), \varepsilon(t)) \mapsto (q(t+h), r(t+h), \varepsilon(t+h))$$

the price paid being the enlarging of the domain by means of a sign space, i.e. to $\mathbb{R}_+^N \times \mathbb{R}_+^N \times \{\pm 1\}^N$.

Another way to avoid the ambiguity is to introduce the variables

$$p_k(t) = \varepsilon_k(t) q_k(t-h) \sqrt{1 - \frac{m_k^2}{q_k^2(t) q_k^2(t-h)}}$$

instead of $r_k(t) = q_k(t-h)$. In the variables (q, p) the map reads

$$q_k^2(t+h) = \frac{m_k^2}{q_k^2(t)} + (f_k(q(t)) - p_k(t))^2$$

$$p_k(t+h) q_k(t+h) = (f_k(q(t)) - p_k(t)) q_k(t) \,.$$

This form of the map has an additional advantage: the variables (q, p) in some cases turn out to be canonically conjugate. More precisely, if $f(q)$ is a gradient of a scalar function, then the map $(q(t), p(t)) \mapsto (q(t+h), p(t+h))$ is symplectic with respect to the standard symplectic structure $\sum_{k=1}^N \mathrm{d}p_k \wedge \mathrm{d}q_k$ (see section 2). The disadvantage of this form is that the natural reversibility of the equation (2) is lost.

In what follows we shall prefer the form (2), always keeping in mind that to every square root $\sqrt{1 - (m_k^2/q_k^2 r_k^2)}$ there is attached a certain sign ε_k.

To conclude this introduction, the continuous limit will be briefly described (for additional remarks concerning the concrete examples, see below). If $m_k = h\mu_k$, $f(q) = 2q + h^2\varphi(q) + \mathrm{o}(h^2)$, then, for small h, equation (2) approximates the ordinary differential equation $\ddot{q}_k = (\mu_k^2/q_k^3) + \varphi_k(q)$. This justifies the title of the present paper.

2. A discrete version of the Wojciechowski system

The remarkable system introduced by Wojciechowski in [6] reads

$$\ddot{q}_k = \dot{p}_k = -\omega_k q_k + \frac{\mu_k^2}{q_k^3} - 2q_k \sum_{j=1}^{N} q_j^2 \qquad 1 \leqslant k \leqslant N .$$

It is Hamiltonian with a Hamiltonian function

$$\mathcal{H}(q,p) = \frac{1}{2}\sum_{k=1}^{N}\left(p_k^2 + \omega_k q_k^2 + \frac{\mu_k^2}{q_k^2}\right) + \frac{1}{2}\left(\sum_{k=1}^{N} q_k^2\right)^2 .$$

For the case when all the ω_k's are different, Wojciechowski found N independent integrals of this system

$$\mathcal{F}_k(q,p) = p_k^2 + \omega_k q_k^2 + \frac{\mu_k^2}{q_k^2} + q_k^2 \sum_{j=1}^{N} q_j^2 + \sum_{j \neq k} \frac{(p_k q_j - p_j q_k)^2 + \frac{\mu_k^2}{q_k^2} q_j^2 + \frac{\mu_j^2}{q_j^2} q_k^2}{\omega_j - \omega_k}$$

and proved that they are in involution. He proposed two different proofs for this fact: one based on a direct and tiresome calculation, and the second following immediately from the Lax pair representations for the Hamiltonian flows with Hamiltonian functions $\mathcal{F}_k$. Such a representation for the flow with the Hamiltonian function $\mathcal{H} = \sum_{k=1}^{N} \mathcal{F}_k$ reads

$$\dot{\mathcal{L}} = [\mathcal{M}, \mathcal{L}]$$

where $\mathcal{L}$, $\mathcal{M}$ are $N+1$ by $N+1$ matrices depending on the phase-space variables p, q and a spectral parameter α

$$\mathcal{L} = \begin{pmatrix} \frac{1}{2}\alpha^2 E + \Omega + qq^{\mathrm{T}} & \alpha q + p + \mathrm{i}\frac{\mu}{q} \\ -\alpha q^{\mathrm{T}} + p^{\mathrm{T}} - \mathrm{i}\left(\frac{\mu}{q}\right)^{\mathrm{T}} & -\frac{1}{2}\alpha^2 - q^{\mathrm{T}} q \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} -\frac{1}{2}\alpha E + \mathrm{i}Q & -q \\ q^{\mathrm{T}} & \frac{1}{2}\alpha \end{pmatrix} .$$

Here E stands for the N by N unity matrix, $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_N)$ and

$$\frac{\mu}{q} = \left(\frac{\mu_1}{q_1}, \ldots, \frac{\mu_N}{q_N}\right)^{\mathrm{T}} \qquad Q = \mathrm{diag}\left(\frac{\mu_1}{q_1^2}, \ldots, \frac{\mu_N}{q_N^2}\right) .$$

The third proof of involutivity could be based on the r-matrix structure for the Lax matrix $\mathcal{L}$, but such a structure was not found in [6]. The corresponding r-matrix turns out to be dependent on the dynamical variables and will be reported elsewhere.

We now present a system of the type (2) which serves as a discrete-time generalization of the Wojciechowski system. Set in (2)

$$f_k(q) = 2c_k q_k \left(1 + \sum_{j=1}^{N} c_j q_j^2\right)^{-1} \qquad 1 \leqslant k \leqslant N \tag{3}$$

where $c_k > 0$ for all k.

Obviously, $f(q)$ is a gradient of a scalar function $\log\left(1 + \sum_{j=1}^{N} c_j q_j^2\right)$. One shows by a direct computation that in such a case the map $(q(t), r(t)) \mapsto (q(t+h), r(t+h))$ preserves the following Poisson bracket on $\mathbb{R}_+^N \times \mathbb{R}_+^N$:

$$\{q_k, q_j\} = \{r_k, r_j\} = 0 \qquad \{q_k, r_j\} = \sqrt{1 - \frac{m_k^2}{q_k^2 r_k^2}}\,\delta_{kj} . \tag{4}$$

It is easy to check that if one denotes

$$p_k = r_k\sqrt{1-\frac{m_k^2}{q_k^2 r_k^2}} \tag{5}$$

then the Poisson bracket (4) preserved by the above-mentioned map takes the canonical shape

$$\{q_k, q_j\} = \{p_k, p_j\} = 0 \qquad \{q_k, p_j\} = \delta_{kj}\,. \tag{6}$$

We now give a Lax pair representation for the system (2), (3). Consider the $N+1$ by $N+1$ matrices depending on the phase variables q, r and a spectral parameter λ:

$$L^+(q,r,\lambda) = \begin{pmatrix} \lambda^{-2}C^{-2} - E + Dqr^{\mathrm{T}} & \lambda Dq - \lambda^{-1}C^{-1}r \\ \lambda^{-1}q^{\mathrm{T}}C^{-1}D^{-1} - \lambda r^{\mathrm{T}} & -\lambda^2 + 1 - q^{\mathrm{T}}D^{-1}r \end{pmatrix}$$

$$L^-(q,r,\lambda) = \begin{pmatrix} \lambda^{-2}C^{-2} - E + qr^{\mathrm{T}}D & \lambda q - \lambda^{-1}D^{-1}C^{-1}r \\ \lambda^{-1}q^{\mathrm{T}}C^{-1} - \lambda r^{\mathrm{T}}D & -\lambda^2 + 1 - q^{\mathrm{T}}D^{-1}r \end{pmatrix}$$

$$M(q,\lambda) = \begin{pmatrix} \lambda^{-1}C^{-1} & -q \\ q^{\mathrm{T}} & \lambda \end{pmatrix}.$$

Here $C = \mathrm{diag}(c_1, \ldots, c_N)$, $D = D(q,r) = \mathrm{diag}(d_1, \ldots, d_N)$ and

$$d_k = \sqrt{1-\frac{m_k^2}{q_k^2 r_k^2}} + \mathrm{i}\frac{m_k}{q_k r_k}$$

so that

$$d_k^{-1} = \sqrt{1-\frac{m_k^2}{q_k^2 r_k^2}} - \mathrm{i}\frac{m_k}{q_k r_k}\,.$$

The matrices $L^{\pm}$ are connected by an obvious similarity transformation

$$L^+(q,r,\lambda)\begin{pmatrix} D(q,r) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} D(q,r) & 0 \\ 0 & 1 \end{pmatrix} L^-(q,r,\lambda)\,. \tag{7}$$

Consider the Lax-like equation

$$M(t,\lambda)L^+(t+h,\lambda) = L^-(t,\lambda)M(t,\lambda) \tag{8}$$

(the dependence of the matrices $L^{\pm}(t,\lambda)$, $M(t,\lambda)$ on t is supposed to appear through the dependence of q, r on t). Because of (7) the equation (8) implies that either of the matrices $L^+(t,\lambda)$, $L^-(t,\lambda)$ evolves in (discrete) time isospectrally. Direct calculation shows that (8) is equivalent to the relations $r(t+h) = q(t)$, $r(t) = q(t-h)$, and

$$\begin{aligned} D(t+h)q(t+h) + D^{-1}(t)q(t-h) &= D^{-1}(t+h)q(t+h) + D(t)q(t-h) \\ &= 2Cq(t)(1+q^{\mathrm{T}}(t)Cq(t))^{-1} \end{aligned} \tag{9}$$

which is nothing other than the system (2), (3).

Hence the characteristic polynomial of either of the matrices $L^{\pm}(\lambda)$ provides us with a number of integrals of this system. One can compute this with the help of the Weinstein–Aronszajn formula (cf [7]), considering the matrices $L^{\pm}(\lambda)$ as rank-3 perturbations of the constant diagonal matrix

$$L_0(\lambda) = \begin{pmatrix} \lambda^{-2}C^{-2} - E & 0 \\ 0 & 1-\lambda^2 \end{pmatrix}.$$

Supposing all the c_k's to be different, one obtains

$$\frac{\det(L^{\pm}(\lambda) - zI)}{\det(L_0(\lambda) - zI)} = 1 - \frac{1}{z - 1 + \lambda^2} \sum_{k=1}^{N} \frac{F_k}{z + 1 - \lambda^{-2} c_k^{-2}}$$

where (up to an additive constant)

$$F_k = c_k^{-1}(q_k^2 + r_k^2) - 2q_k r_k \sqrt{1 - \frac{m_k^2}{q_k^2 r_k^2}} + q_k^2 r_k^2 + \sum_{j \neq k} \frac{c_k c_j}{c_k^2 - c_j^2} (q_k^2 r_j^2 + r_k^2 q_j^2)$$

$$- \sum_{j \neq k} \frac{2c_j^2}{c_k^2 - c_j^2} q_k r_k q_j r_j \sqrt{1 - \frac{m_k^2}{q_k^2 r_k^2}} \sqrt{1 - \frac{m_j^2}{q_j^2 r_j^2}} \tag{10}$$

The involutivity of these integrals with respect to the Poisson bracket (4) may be verified by direct calculation, like that presented in [6]. Unfortunately, we did not succeed in performing a proof based on either of the two alternative approaches mentioned above (Lax representations for the F_k flows and r-matrix). Let us note in this connection that in the canonically conjugate variables (q, p) (see equations (5), (6)) the Lax matrix $L^-(\lambda)$ and the integrals F_k are rational, which simplifies calculations considerably:

$$L^- = \begin{pmatrix} \lambda^{-2} C^{-2} - E + q \left(p + \mathrm{i}\frac{m}{q}\right)^{\mathrm{T}} & \lambda q - \lambda^{-1} C^{-1} \left(p - \mathrm{i}\frac{m}{q}\right) \\ \lambda^{-1} q^{\mathrm{T}} C^{-1} - \lambda \left(p + \mathrm{i}\dfrac{m}{q}\right)^{\mathrm{T}} & -\lambda^2 + 1 - q^{\mathrm{T}} \left(p - \mathrm{i}\frac{m}{q}\right) \end{pmatrix}$$

$$F_k = c_k^{-1} \left(q_k^2 + p_k^2 + \frac{m_k^2}{q_k^2}\right) - 2q_k p_k + q_k^2 p_k^2$$

$$+ \sum_{j \neq k} \frac{c_k c_j}{c_k^2 - c_j^2} \left(q_k^2 p_j^2 + p_k^2 q_j^2 + \frac{m_k^2}{q_k^2} q_j^2 + q_k^2 \frac{m_j^2}{q_j^2}\right) - \sum_{j \neq k} \frac{2c_j^2}{c_k^2 - c_j^2} q_k p_k q_j p_j$$

(up to an additive constant).

We conclude this section with the following remark on the continuous limit. To recover the Wojciechowski system from (2), (3) in the limit $h \to 0$ one has to perform the change of variables $q \mapsto hq$, and assume $m_k = h^3 \mu_k$, $c_k = 1 - \frac{1}{2} h^2 \omega_k + o(h^2)$. The spectral parameters of the discrete- and continuous-time Lax pairs should be connected by $\lambda = 1 + \frac{1}{2} h\alpha + o(h)$.

3. A discrete version of the Rosochatius system

The system introduced by Rosochatius in the 19th century and then incorporated in the modern framework of integrable dynamical systems in [7–11] is described by the equations

$$\ddot{q}_k = \dot{p}_k = -\omega_k q_k + \frac{\mu_k^2}{q_k^3} - \Lambda q_k \qquad 1 \leqslant k \leqslant N$$

where the Lagrange multiplier Λ must be chosen to assure that the point $q = (q_1, \ldots, q_N)^{\mathrm{T}}$ remains on the unit sphere $S^{N-1} \subset \mathbb{R}^N$: $\langle q, q \rangle = 1$, $\langle q, p \rangle = 0$, i.e.

$$\Lambda = \sum_{j=1}^{N} \left(p_j^2 - \omega_j q_j^2 + \frac{\mu_j^2}{q_j^2}\right).$$

This system is Hamiltonian with a Hamiltonian function

$$\mathcal{H}(q,p)=\tfrac{1}{2}\sum_{k=1}^{N}\left(p_k^2+\omega_k q_k^2+\frac{\mu_k^2}{q_k^2}\right)$$

with respect to the Poison bracket $\{\,,\}^*$ which is a Dirac reduction of the standard Poisson bracket to the common level set of the two functions $\varphi_1=\langle q,q\rangle-1=0$, $\varphi_2=\langle q,p\rangle=0$. Recall the explicit formula for the Dirac reduction [7]:

$$\{\mathcal{F},\mathcal{G}\}^*=\{\mathcal{F},\mathcal{G}\}+\frac{\{\mathcal{F},\varphi_2\}\{\mathcal{G},\varphi_1\}-\{\mathcal{F},\varphi_1\}\{\mathcal{G},\varphi_2\}}{\{\varphi_1,\varphi_2\}}$$

i.e. in our case

$$\{q_k,q_j\}^*=0\qquad \{p_k,q_j\}^*=\delta_{kj}-q_kq_j\qquad \{p_k,p_j\}^*=q_kp_j-p_kq_j\,.$$

For the case when all the ω_k's are different Moser gave the expressions for N integrals of this system,

$$\mathcal{F}_k(q,p)=q_k^2+\sum_{j\neq k}\frac{(p_kq_j-p_jq_k)^2+\frac{\mu_k^2}{q_k^2}q_j^2+\frac{\mu_j^2}{q_j^2}q_k^2}{\omega_k-\omega_j}$$

which, however, are not independent on the constrained phase space since $\sum_{k=1}^{N}\mathcal{F}_k=\langle q,q\rangle=1$. The integrals are in involution with respect to both the unreduced and the reduced Poisson brackets, which was proved in [7] by means of construction Lax pair representations for all Hamiltonian flows generated by $\mathcal{F}_k$. Such a representation for the $\mathcal{H}=\sum_{k=1}^{N}\omega_k\mathcal{F}_k$ flow reads

$$\dot{\mathcal{L}}=[\mathcal{M},\mathcal{L}]$$

where $\mathcal{L}$, $\mathcal{M}$ are N by N matrices depending on the phase-space variables p,q and a spectral parameter α:

$$\mathcal{L}=-\Omega+\alpha\left(pq^{\mathrm{T}}-qp^{\mathrm{T}}+\mathrm{i}\left(\frac{\mu}{q}q^{\mathrm{T}}+q\left(\frac{\mu}{q}\right)^{\mathrm{T}}\right)\right)+\alpha^2qq^{\mathrm{T}}\qquad \mathcal{M}=\alpha qq^{\mathrm{T}}+\mathrm{i}Q$$

(the meaning of the symbols Ω, μ/q, and Q is the same as in section 2).

We now present a system of the type (2) (but with $f_k(q(t),q(t-h))$ on the right-hand side) which is a discrete-time generalization of the Rosochatius system: for this system

$$f_k(q,r)=\Lambda(q,r)c_kq_k \tag{11}$$

where a Lagrange multiplier $\Lambda(q(t),q(t-h))$ must be chosen in such a way as to ensure that $q(t+h)$ lies on S^{N-1} provided that $q(t)$ and $q(t-h)$ both do likewise. It is easy to obtain

$$\Lambda(q,r)=2\left(\sum_{k=1}^{N}c_kq_kr_k\sqrt{1-\frac{m_k^2}{q_k^2r_k^2}}\right)\Big/\left(\sum_{k=1}^{N}c_k^2q_k^2\right). \tag{12}$$

One obtains immediately from (2), (11) that the choice (12) implies

$$\sum_{k=1}^{N}c_kq_k(t)q_k(t-h)\sqrt{1-\frac{m_k^2}{q_k^2(t)q_k^2(t-h)}}=\sum_{k=1}^{N}c_kq_k(t+h)q_k(t)\sqrt{1-\frac{m_k^2}{q_k^2(t+h)q_k^2(t)}}$$

which makes obvious the reversibility of our system.

The equation (2) with the choice (11), (12) defines a map

$$(q(t),r(t))\mapsto(q(t+h),r(t+h))$$

of $S^{N-1}\times S^{N-1}$ onto itself. This map is symplectic with respect to the Dirac reduction to this set (described by $\varphi_1=\langle q,q\rangle-1=0$, $\varphi_2=\langle r,r\rangle-1=0$) of the following Poisson bracket:

$$\{q_k,q_j\}=\{r_k,r_j\}=0 \qquad \{q_k,r_j\}=c_k\sqrt{1-\frac{m_k^2}{q_k^2r_k^2}}\delta_{kj}\,. \tag{13}$$

This fact can be proved by a direct calculation based on the explicit formulas for the reduced bracket

$$\begin{aligned}&\{q_k,q_j\}^*=\{r_k,r_j\}^*=0\\&\{q_k,r_j\}^*=c_k\sqrt{1-\frac{m_k^2}{q_k^2r_k^2}}\delta_{kj}-\frac{c_kc_jr_kq_j\sqrt{1-\frac{m_k^2}{q_k^2r_k^2}}\sqrt{1-\frac{m_j^2}{q_j^2r_j^2}}}{\sum_{l=1}^N c_lq_lr_l\sqrt{1-\frac{m_l^2}{q_l^2r_l^2}}}\,.\end{aligned} \tag{14}$$

A more conceptual proof of this fact can be obtained along the lines of [3], if one represents (2), (11) as a discrete-time Lagrangean system on $S^{N-1}\times S^{N-1}$ with a Lagrangian $T(q,r)$ defined by the relations

$$\frac{\partial T}{\partial q_k}=c_k^{-1}r_k\sqrt{1-\frac{m_k^2}{q_k^2r_k^2}} \qquad \frac{\partial T}{\partial r_k}=c_k^{-1}q_k\sqrt{1-\frac{m_k^2}{q_k^2r_k^2}}\,.$$

The Poisson bracket (13) is generated by the symplectic form

$$\sum_{k,j=1}^N\frac{\partial^2T}{\partial q_k\partial r_j}\mathrm{d}q_k\wedge\mathrm{d}r_j=\sum_{k=1}^N\left(c_k\sqrt{1-\frac{m_k^2}{q_k^2r_k^2}}\right)^{-1}\mathrm{d}q_k\wedge\mathrm{d}r_k$$

on $\mathbb{R}^N\times\mathbb{R}^N$, and the Poisson bracket (14) is generated by its restriction on $S^{N-1}\times S^{N-1}$.

We now give a Lax pair representation for the system (2), (11), (12). Consider two N by N matrices depending on the phase variables (q,r) and a spectral parameter λ:

$$L(q,r,\lambda)=C^{-2}+\lambda(rq^{\mathrm T}C^{-1}D^{-1}-DC^{-1}qr^{\mathrm T})-\lambda^2rr^{\mathrm T} \tag{15}$$

$$M(q,r,\lambda)=C^{-1}D^{-1}-\lambda qr^{\mathrm T} \tag{16}$$

(the symbols C, D have the same meaning as in section 2). Then it is not hard to see that the matrix equation

$$L(t+h,\lambda)M(t,\lambda)=M(t,\lambda)L(t,\lambda) \tag{17}$$

on $S^{N-1}\times S^{N-1}$ is equivalent to $r(t+h)=q(t)$, $r(t)=q(t-h)$, and

$$\begin{aligned}&C^{-1}[D(t+h)q(t+h)+D^{-1}(t)q(t-h)]q^{\mathrm T}(t)\\&\quad=q(t)[q^{\mathrm T}(t+h)D^{-1}(t+h)+q^{\mathrm T}(t-h)D(t)]C^{-1}\end{aligned} \tag{18}$$

which is just equvalent to (12).

Note also the following factorization property:

$$L(q,r,\lambda)=\widetilde{M}(q,r,\lambda)M(q,r,\lambda) \tag{19}$$

where

$$\widetilde{M}(q,r,\lambda)=C^{-1}D+\lambda rq^{\mathrm T}\,. \tag{20}$$

Now we have from (17), (19)

$$L(t,\lambda)=\widetilde{M}(t,\lambda)M(t,\lambda) \qquad L(t+h,\lambda)=M(t,\lambda)\widetilde{M}(t,\lambda) \tag{21}$$

hence the machinery of matrix factorization developed in [3] also applies for the discrete-time Rosochatius system.

Applying the Weinsteinstein–Aronszajn formula, we obtain

$$\frac{\det(L(\lambda)-zI)}{\det(C^{-2}-zI)}=1+\lambda^2\sum_{k=1}^{N}\frac{F_k}{z-c_k^{-2}}$$

where, up to an additive constant,

$$F_k=r_k^2+\sum_{j\neq k}\frac{c_j^2}{c_j^2-c_k^2}q_k^2r_j^2+\sum_{j\neq k}\frac{c_k^2}{c_j^2-c_k^2}r_k^2q_j^2$$

$$-2\sum_{j\neq k}\frac{c_kc_j}{c_j^2-c_k^2}q_kr_kq_jr_j\sqrt{1-\frac{m_k^2}{q_k^2r_k^2}}\sqrt{1-\frac{m_j^2}{q_j^2r_j^2}}\,. \tag{22}$$

Note that this time the variables (q,p) (see equation (5)) are not canonically conjugate in the strict sence, since the unreduced Poisson bracket (13) in these coordinates looks like

$$\{q_k,q_j\}=\{p_k,p_j\}=0 \qquad \{q_k,p_j\}=c_k\delta_{kj}$$

and the reduced one (14) looks like

$$\{q_k,q_j\}^*=0 \qquad \{p_k,p_j\}^*=\frac{c_kc_j}{\langle Cq,p\rangle}\left(q_k\frac{m_j^2}{q_j^3}-\frac{m_k^2}{q_k^3}q_j\right)$$

$$\{q_k,p_j\}^*=c_k\delta_{kj}-\frac{c_kc_jp_kq_j}{\langle Cq,p\rangle}\,.$$

Still, these expressions are simple and useful in calculations. The matrix

$$D^{-1}LD=C^{-2}+\lambda\left(\left(p-\mathrm{i}\frac{m}{q}\right)q^{\mathrm{T}}C^{-1}-C^{-1}q\left(p+\mathrm{i}\frac{m}{q}\right)^{\mathrm{T}}\right)$$

$$-\lambda^2\left(p-\mathrm{i}\frac{m}{q}\right)\left(p+\mathrm{i}\frac{m}{q}\right)^{\mathrm{T}}$$

and the integrals

$$F_k=p_k^2+\frac{m_k^2}{q_k^2}+\sum_{j\neq k}\frac{(q_kc_jp_j-c_kp_kq_j)^2+q_k^2\frac{c_j^2m_j^2}{q_j^2}+\frac{c_k^2m_k^2}{q_k^2}q_j^2}{c_j^2-c_k^2}$$

turn out to be rational in the variables (q,p), just as in section 2.

The only proof of involutivity of these integrals we have at present is based on direct calculation, which is omitted here for the sake of brevity.

To conclude this section, note that the continuous limit in this case is achieved if one sets $m_k=h\mu_k$, $c_k=1-\frac{1}{2}h^2\omega_k+\mathrm{o}(h^2)$ and lets $h\to 0$; the spectral parameters in the continuous- and discrete-time Lax pairs should be connected by $\lambda=h\alpha$.

4. Concluding remarks

In this paper we introduce a novel class of integrable symplectic maps which, in the continuous limit, turn into well known nonlinear oscillators subject to the additional inverse-square potential. The Lax pair representations for these maps are found, which provide a systematic way of finding the integrals.

Conceptual proof of involutivity of these integrals, however, is still lacking at the moment. It would be rather interesting to find out the r-matrix structure (which we expect to be dependent on the dynamical variables) for our models.

Note further that both the Wojciechowski and the Rosochatius systems can be solved by means of the method of variables separation. Performing its discrete-time analogue is an important problem that remains open.

Finally, connection with the theory of Adams, Harnad and Previato [11] remains to be clarified.

Work on these issues is now in progress.

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